A GENERAL CLASS OF LINEAR TRANSFORMATIONS OF WIENER INTEGRALS([1,2])

BY

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Introduction. Let C be the space of continuous functions $x(t)$, $0 \leqq t \leqq 1$, where $x(0)=0$. We consider one to one linear transformations of C onto C of the form

$$(0.1) \qquad y(t) = x(t) + \int_0^1 L(t, s)dx(s).$$

N. Wiener [4] has given an integral on C. We obtain the transformation formula for this integral under transformations of type (0.1). The transformations of type (0.1) which we consider contain as a special case the transformations considered by Cameron and Martin [1].

Let I be the interval $[0, 1]$ and I^2 the square $I \otimes I$. If $x \in C$, we define

$$|||x||| = \max_{t \in I} | x(t) | .$$

If $K(t, s)$ is bounded and measurable on I^2 and $K(t, t)$ is measurable on I, we denote the Fredholm determinant of $K(t, s)$ evaluated at $\lambda = -1$ by

$$(0.2) \quad D(K) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 \cdots \int_0^1 \begin{vmatrix} K(s_1, s_1) & \cdots & K(s_1, s_n) \\ \vdots & & \vdots \\ K(s_n, s_1) & \cdots & K(s_n, s_n) \end{vmatrix} ds_1 \cdots ds_n.$$

We say that $M(t, s)$ is of bounded variation (B.V.) on I^2 if there exists $(t_0, s_0) \in I^2$ such that $M(t_0, s)$ and $M(t, s_0)$ are of B.V. on I and

$$\operatorname*{var}_{(t,s) \in I^2} M(t, s) < \infty.$$

Here

$$(0.3) \quad \begin{aligned} &\operatorname*{var}_{(t,s) \in I^2} M(t, s) \\ &\quad = \sup \sum_{j=1}^{m} \sum_{i=1}^{n} | M(t_i, s_j) - M(t_i, s_{j-1}) + M(t_{i-1}, s_{j-1}) - M(t_{i-1}, s_j) | , \end{aligned}$$

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where the supremum is taken over all partitions $0 \leqq t_0 \leqq \cdots \leqq t_n \leqq 1$ and $0 \leqq s_0 \leqq \cdots \leqq s_m \leqq 1$ of I. The following theorem is the main result of this paper.

THEOREM A. *Let $L(t, s)$ satisfy the following conditions*

(0.4) $L(0, s) = 0$, $s \in I$.

(0.5) *$L(t, s)$ is absolutely continuous in t on I for each $s \in I$ and there exist $M(t, s)$ and $J(t)$ such that*

$$\frac{\partial}{\partial t} L(t, s) = \overline{M}(t, s) \tag{0.6}$$

for almost all t on I for all $s \in I$, where

$$\overline{M}(t, s) = \begin{cases} M(t, s), & 0 \leqq t < s \leqq 1, \\ M(t, s) + \dfrac{1}{2} J(t), & 0 \leqq t = s \leqq 1, \\ M(t, s) + J(t), & 0 \leqq s < t \leqq 1, \end{cases} \tag{0.7}$$

and where

(0.8) *$M(t, s)$ is of B.V. on I^2 and $J(t)$ is of B.V. on I. Also*

$$D(\overline{M}) \neq 0. \tag{0.9}$$

Then the transformation (0.1) *carries C onto C in a one to one manner and if $F(y)$ is a measurable functional such that either side of the following equation exists, both sides exist and are equal.*

$$\int_C^w F(y) d_w y = | D(\overline{M}) | \int_C^w F\left(x(\cdot) + \int_0^1 L(\cdot, s) dx(s)\right) \exp\{-\Psi(x)\} d_w x \tag{0.10}$$

where

$$\begin{aligned} \Psi(x) &= \int_0^1 \left[\int_0^1 \overline{M}(t, s) dx(s) \right]^2 dt + 2 \int_0^1 \int_0^1 M(t, s) dx(s) dx(t) \\ &\quad + \int_0^1 J(t) d(x(t))^2. \\ &= \int_0^1 \int_0^1 \left\{ \overline{M}(t, s) + \overline{M}(s, t) + \int_0^1 \overline{M}(t, u) \overline{M}(s, u) du \right\} dx(s) dx(t). \end{aligned} \tag{0.11}$$

We prove Theorem A by starting with Theorem B, a theorem of Cameron and Martin [1], which appears below in a slightly weaker form. From Theorem B we develop Theorem C to obtain Theorem A.

THEOREM B. *Let $H(t, s)$ be measurable on I^2 and*

$$\operatorname*{var}_{t\in I} H(t, s) < B < \infty \tag{0.12}$$

independent of s. Assume also that

(0.13) *$J(t)$ is continuous and of B.V. on I,*

(0.14) *$H(t, s)$ is continuous in s on I for every $t\in I$, and*

(0.15) *$D(K)\neq 0$ where*

$$K(t, s) = \begin{cases} \int_0^t H(u, s)du, & 0 \leqq t < s \leqq 1, \\ \int_0^t H(u, t)du + \frac{1}{2}J(t), & 0 \leqq t = s \leqq 1, \\ \int_0^t H(u, s)du + J(s), & 0 \leqq s < t \leqq 1. \end{cases} \tag{0.16}$$

Then if $F(y)$ is a measurable functional such that either side of the following equation exists, both sides exist and are equal.

$$\begin{aligned} \int_c^w F(y)d_wy = | D(K) | \int_c^w F\Big(x(\cdot) &+ \int_0^1 \int_0^{(\cdot)} H(u, s)dux(s)ds \\ &+ \int_0^{(\cdot)} J(s)x(s)ds\Big) \exp\{-\Phi(x)\} d_wx \end{aligned} \tag{0.17}$$

where

$$\begin{aligned} \Phi(x) = \int_0^1 &\left\{\int_0^1 H(t, s)x(s)ds + J(t)x(t)\right\}^2 dt \\ &+ 2\int_0^1 \int_0^1 H(t, s)x(s)dsdx(t) + \int_0^1 J(t)d(x(t))^2. \end{aligned} \tag{0.18}$$

THEOREM C. *Let $M(t, s)$ and $J(t)$ satisfy* (0.7)–(0.9), *let $M(t, s)$ be continuous in s on I for each $t\in I$, and let*

$$J(1) = M(t, 1) = M(1, t) = 0, \qquad t \in I. \tag{0.19}$$

Let $\epsilon > 0$ and define

$$M^\epsilon(t, s) = \frac{1}{\epsilon}\int_s^{s+\epsilon} M(t, v)dv, \qquad (t, s) \in I^2, \tag{0.20}$$

where $M(t, v) = 0$ for $v > 1$;

$$J^\epsilon(t) = \frac{1}{\epsilon}\int_t^{t+\epsilon} J(u)du, \qquad t \in I, \tag{0.21}$$

where $J(u) = 0$ for $u > 1$; and

$$(0.22)\qquad (M^\epsilon)^*(t,s) = \begin{cases} M^\epsilon(t,s), & 0 \leqq t < s \leqq 1, \\ M^\epsilon(t,s) + \dfrac{1}{2} J^\epsilon(t), & 0 \leqq s = t \leqq 1, \\ M^\epsilon(t,s) + J^\epsilon(t), & 0 \leqq s < t \leqq 1. \end{cases}$$

Then there exists $\delta>0$ such that if $0<\epsilon<\delta$ and $F(y)$ is a measurable functional such that either side of the following equation exists, both sides exist and are equal.

$$(0.23)\qquad \begin{aligned} \int_c^w F(y)d_wy = {} & \left| D(M^\epsilon)^* \right| \int_c^w F\left(x(\cdot) + \int_0^1 \int_0^{(\cdot)} (M\epsilon)^*(u,s)dudx(s)\right) \\ & \cdot \exp\left\{ - \int_0^1 \left[\int_0^1 (M^\epsilon)^*(t,s)dx(s)\right]^2 dt \right. \\ & \left. - 2\int_0^1 \int_0^1 M^\epsilon(t,s)dx(s)dx(t) - \int_0^1 J^\epsilon(t)d(x(t))^2 \right\} d_wx. \end{aligned}$$

1. In this section we give some lemmas used in the proof of Theorem C.

LEMMA 1. *Let $N_\epsilon(t, s)$ be measurable on I^2 and $N_\epsilon(t, t)$ be measurable on I for all $\epsilon>0$. Let $N_\epsilon(t, s)$ be bounded on I^2 independent of $\epsilon>0$. Let*

$$\lim_{\epsilon\to 0^+} N_\epsilon(t,s) = N(t,s)$$

almost everywhere on I^2 and

$$\lim_{\epsilon\to 0^+} N_\epsilon(t,t) = N(t,t)$$

almost everywhere on I. Then

$$(1.1)\qquad \lim_{\epsilon\to 0^+} D(N_\epsilon) = D(N).$$

Proof. Let $\eta>0$ be given. By application of Hadamard's lemma [3], it follows that there exists an integer K such that

$$\left| D(N_\epsilon) - \sum_{n=0}^{K} \frac{1}{n!} \int_0^1 \cdots \int_0^1 \begin{vmatrix} N_\epsilon(s_1,s_1) & \cdots & N_\epsilon(s_1,s_n) \\ \vdots & & \vdots \\ N_\epsilon(s_n,s_1) & \cdots & N_\epsilon(s_n,s_n) \end{vmatrix} ds_1 \cdots ds_n \right| < \frac{\eta}{3}$$

independent of $\epsilon>0$ and

$$\left| D(N) - \sum_{n=0}^{K} \frac{1}{n!} \int_0^1 \cdots \int_0^1 \begin{vmatrix} N(s_1,s_1) & \cdots & N(s_1,s_n) \\ \vdots & & \vdots \\ N(s_n,s_1) & \cdots & N(s_n,s_n) \end{vmatrix} ds_1 \cdots ds_n \right| < \frac{\eta}{3}\cdot$$

It follows immediately from the hypothesis and bounded convergence that

$$\lim_{\epsilon\to 0^+}\sum_{n=1}^{K}\frac{1}{n!}\int_0^1\cdots\int_0^1\begin{vmatrix} N_\epsilon(s_1,s_1) & \cdots & N_\epsilon(s_1,s_n)\\ \vdots & & \vdots\\ N_\epsilon(s_n,s_1) & \cdots & N_\epsilon(s_n,s_n)\end{vmatrix}ds_1\cdots ds_n$$

$$=\sum_{n=1}^{K}\frac{1}{n!}\int_0^1\cdots\int_0^1\begin{vmatrix} N(s_1,s_1) & \cdots & N(s_1,s_n)\\ \vdots & & \vdots\\ N(s_n,s_1) & \cdots & N(s_n,s_n)\end{vmatrix}ds_1\cdots ds_n.$$

Therefore there exists $\delta>0$ such that $0<\epsilon<\delta$ implies

$$|D(N_\epsilon)-D(N)|<\eta.$$

This implies (1.1).

The following result was communicated to the author by R. H. Cameron.

LEMMA 2. *Let $N(t, s)$ be bounded and measurable on I^2. Then*

$$D\left(\int_0^t N(u,s)ds\right)=D\left(\int_s^1 N(t,v)dv\right). \tag{1.2}$$

Proof. A typical term from the expansion of a determinant in the expansion (0.2) of $D(\int_0^t N(u, s)du)$ can be written

$$\begin{aligned}
&\int_0^1\cdots\int_0^1 ds_1\cdots ds_n\prod_{i=1}^{m}\int_0^{s_i}N(u_i,s_i)du_i\int_0^{s_{m+1}}N(u_{m+1},s_{m+2})du_{m+1}\\
&\qquad\cdots\int_0^{s_{n-1}}N(u_{n-1},s_n)du_{n-1}\int_0^{s_n}N(u_n,s_{m+1})du_n\\
&=\int_0^1\cdots\int_0^1 du_1\cdots du_n\prod_{i=1}^{m}\int_{u_i}^1 N(u_i,s_i)ds_i\int_{u_{m+1}}^1 N(u_n,s_{m+1})ds_{m+1}\\
&\qquad\cdot\int_{u_{m+2}}^1 N(u_{m+1},s_{m+2})ds_{m+2}\cdots\int_{u_n}^1 N(u_{n-1},s_n)ds_n\\
&=\int_0^1\cdots\int_0^1 ds_1\cdots ds_n\prod_{i=1}^{m}\int_{s_i}^1 N(s_i,v_i)dv_i\int_{s_{m+1}}^1 N(s_n,v_{m+1})dv_{m+1}\\
&\qquad\cdot\int_{s_{m+2}}^1 N(s_{m+1},v_{m+2})dv_{m+2}\cdots\int_{s_n}^1 N(s_{n-1},v_n)dv_n\\
&=\int_0^1\cdots\int_0^1 ds_1\cdots ds_n\prod_{i=1}^{m}\int_{s_i}^1 N(s_i,v_i)dv_i\int_{s_{m+2}}^1 N(s_{m+1},v_{m+2})dv_{m+2}\\
&\qquad\cdots\int_{s_n}^1 N(s_{n-1},v_n)dv_n\int_{s_{m+1}}^1 N(s_n,v_{m+1})dv_{m+1}
\end{aligned} \tag{1.3}$$

by use of the Fubini theorem, a change of dummy variables, and a change in the order of multiplication. The extreme right hand member of (1.3) is the same corresponding term in the expansion of

$$D\left(\int_s^1 N(t, v)dv\right)$$

as the extreme left hand member of (1.3) is in the expansion of

$$D\left(\int_0^t N(u, s)du\right).$$

This establishes (1.2).

LEMMA 3. *Let $M(t, s)$ be bounded and measurable on I^2 and let $J(t)$ be bounded and measurable on I. Let $J^\epsilon(t)$, $M^\epsilon(t, s)$, and $(M^\epsilon)^*(t, s)$ be defined as in* (0.20), (0.21) *and* (0.22) *and*

$$(1.4)\quad K_\epsilon(t, s) = \begin{cases} -\int_0^t \frac{1}{\epsilon}[M(u, s+\epsilon) - M(u, s)]du, & 0 \leqq t < s \leqq 1, \\ -\int_0^t \frac{1}{\epsilon}[M(u, s+\epsilon) - M(u, s)]du + \frac{1}{2}J^\epsilon(s), & 0 \leqq t = s \leqq 1, \\ -\int_0^t \frac{1}{\epsilon}[M(u, s+\epsilon) - M(u, s)]du + J^\epsilon(s), & 0 \leqq s < t \leqq 1. \end{cases}$$

Then for all $\epsilon > 0$,

$$(1.5)\qquad D(K_\epsilon) = D(M^\epsilon)^*.$$

Proof. Let

$$N(t, s) = \frac{1}{\epsilon}[M(t, s+\epsilon) - M(t, s)], \qquad \eta > 0,$$

$$\phi_\eta(u) = \begin{cases} 0, & u \leqq -\eta, \\ \frac{1}{\eta^2}(u+\eta), & -\eta \leqq u \leqq 0, \\ -\frac{1}{\eta^2}(u-\eta), & 0 \leqq u \leqq \eta, \\ 0, & u \geqq \eta, \end{cases}$$

and

$$N_\eta(t, s) = -N(t, s) + \phi_\eta(t - s)J^\epsilon(s).$$

Then the following two equations hold insofar as their right hand sides are defined.

$$\int_0^t N_\eta(u, s)du = -\int_0^t N(u, s)du + J^\epsilon(s)\begin{cases} 0, & \eta < t + \eta < s < 1, \\ \frac{1}{2}, & \eta < t = s < 1, \\ 1, & 2\eta < s + \eta < t < 1, \end{cases}$$

and

$$\int_s^1 N_\eta(t, u)du = -\int_s^1 N(t, u)du$$

$$+\begin{cases} 0, & 0 < t < s - \eta < 1 - \eta, \\ \int_t^{t+\eta} \frac{1}{\eta^2}(t - u + \eta)J^\epsilon(u)du, & 0 < t = s < 1 - \eta, \\ \int_t^{t+\eta} \frac{1}{\eta^2}(t - u + \eta)J^\epsilon(u)du - \int_{t-\eta}^t \frac{1}{\eta^2}(t - u - \eta)J^\epsilon(u)du, & \\ & 0 < s < t - \eta < 1 - 2\eta. \end{cases}$$

Since $J^\epsilon(t)$ is continuous on I, we have

$$\begin{aligned} \lim_{\eta\to 0^+}\int_0^t N_\eta(u, s)du &= -\int_0^t N(u, s)dy + J^\epsilon(s)\begin{cases} 0, & 0 < t < s < 1, \\ \frac{1}{2}, & 0 < s = t < 1, \\ 1, & 0 < s < t < 1, \end{cases} \\ &= K_\epsilon(t, s) \end{aligned} \tag{1.6}$$

and

$$\lim_{\eta\to 0^+}\int_s^1 N_\eta(t, u)du = -\int_s^1 N(t, u)du + J^\epsilon(t)\begin{cases} 0, & 0 < t < 2 < 1, \\ \frac{1}{2}, & 0 < t = s < 1, \\ 1, & 0 < s < t < 1. \end{cases} \tag{1.7}$$

Since $N_\eta(t, s)$ is bounded and measurable on I^2 for every $\eta > 0$, we use Lemma 2 to obtain

$$D\left(\int_0^t N_\eta(u, s)du\right) = D\left(\int_s^1 N_\eta(t, v)dv\right). \tag{1.8}$$

Since

$$\lim_{\eta\to 0^+}\int_s^1 N_\eta(t, v)dv$$

exists almost everywhere on I^2,

$$\lim_{\eta\to 0^+}\int_t^1 N_\eta(t, v)dv$$

exists almost everywhere on I,

$$\int_s^1 N_\eta(t, v)dv$$

is measurable on I^2 by the Fubini theorem in three dimensions,

$$\int_t^1 N_\eta(t, v)dv$$

is measurable on I by the Fubini theorem, and

$$\int_s^1 N_\eta(t, v)dv$$

is bounded independent of $\eta>0$, we can use Lemma 1 to obtain

$$\lim_{\eta\to 0^+} D\left(\int_s^1 N_\eta(t, v)dv\right) = D\left(\lim_{\eta\to 0^+}\int_s^1 N_\eta(t, v)dv\right). \tag{1.9}$$

Similarly,

$$\lim_{\eta\to 0^+} D\left(\int_0^t N_\eta(u, s)du\right) = D\left(\lim_{\eta\to 0^+}\int_0^t N_\eta(u, s)du\right). \tag{1.10}$$

Since

$$\begin{aligned} -\int_s^1 N(t, v)dv &= -\int_s^1 \frac{1}{\epsilon}\,[M(t, v+\epsilon) - M(t, v)]dv \\ &= -\int_{s+\epsilon}^1 \frac{1}{\epsilon} M(t, v)dv + \int_s^1 \frac{1}{\epsilon} M(t, v)dv \\ &= \frac{1}{\epsilon}\int_s^{s+\epsilon} M(t, v)dv = M^\epsilon(t, s), \end{aligned}$$

we have from (1.7),

$$\lim_{\eta\to 0^+}\int_s^1 N_\eta(t, v)dv = (M^\epsilon)^*(t, s). \tag{1.11}$$

Therefore (1.6), (1.8), (1.9), (1.10), and (1.11) imply (1.5).

LEMMA 4. *Let $M(t, s)$ be of B.V. on I^2. Then if $(t_1, s_1)\in I^2$, $M(t, s_1)$ and $M(t_1, s)$ are of B.V. on I. Also $M(t, s)$ is measurable on I^2, $M(t, t)$ is measurable on I, and for almost every $t\in I$,*

$$\lim_{s \to t^+} M(t, s) = M(t, t). \tag{1.12}$$

Furthermore, if $\epsilon > 0$ *and* $M^\epsilon(t, s)$ *is defined by* (0.20), *then there exists* $B < \infty$ *independent of* $\epsilon > 0$ *such that*

$$\operatorname*{var}_{(t,s) \in I^2} M^\epsilon(t, s) < B. \tag{1.13}$$

Proof. Let $(t_1, s_1) \in I^2$. Let $(t_0, s_0) \in I^2$ be such that $M(t, s_0)$ and $M(t_0, s)$ are of B.V. on I and $0 \leqq u_0 \leqq \cdots \leqq u_n \leqq 1$ be a partition of I. Then

$$\begin{aligned} \sum_{i=1}^n |M(u_i, s_1) - M(u_{i-1}, s_1)| \\ &\leqq \sum_{i=1}^n | M(u_i, s_1) - M(u_{i-1}, s_1) - M(u_i, s_0) + M(u_{i-1}, s_0) | \\ &\quad + \sum_{i=1}^n | M(u_i, s_0) - M(u_{i-1}, s_0) | . \end{aligned}$$

Since the right hand side is bounded independent of the partition $0 \leqq u_0 \leqq \cdots \leqq u_n \leqq 1$, $M(t, s_1)$ is of B.V. on I. Similarly, $M(t_1, s)$ is of B.V. on I.

Let $(t, s) \in I^2$ and $\{0 \leqq t_0 \leqq \cdots \leqq t_n \leqq t;\ 0 \leqq s_0 \leqq \cdots \leqq s_m < s\}$ be a partition of $[0, t] \otimes [0, s]$. Let

$$P(t, s) = \sup \sum [M(t_i, s_j) - M(t_{i-1}, s_j) - M(t_i, s_{j-1}) + M(t_{i-1}, s_{j-1})],$$
$$N(t, s) = \sup \sum \{-[M(t_i, s_j) - M(t_{i-1}, s_j) - M(t_i, s_{j-1}) + M(t_{i-1}, s_{j-1})]\},$$

where the sums are taken over the positive terms only and the suprema are taken over all partitions of $[0, t] \otimes [0, s]$. It is easy to see that

$$M(t, s) = P(t, s) - N(t, s) + M(0, s) + M(t, 0) - M(0, 0) \tag{1.14}$$

and that if $0 \leqq u \leqq v \leqq 1$ and $t \in I$, then

$$P(u, u) \leqq P(v, v), \tag{1.15}$$

$$P(t, u) \leqq P(t, v), \text{ and} \tag{1.16}$$

$$P(u, t) \leqq P(v, t). \tag{1.17}$$

We will show that $P(t, s)$ is measurable on I^2. Let $\alpha > 0$. Then

$$\phi(t) = \sup_{s \in I} \{s: P(t, s) < \alpha\}, \qquad t \in I,$$

is a monotone decreasing function because if $t' < t''$ and $\phi(t') < \phi(t'')$, then there exists s'' such that $\phi(t') < s'' < \phi(t'')$ and according to (1.17), $P(t', s'') \leqq P(t'', s'') < \alpha$ so that $\phi(t') \geqq s''$, a contradiction. Therefore $\phi(t)$ is measurable and its ordinate set

$$\{(t, s) \in I^2: P(t, s) < \alpha\}$$

is measurable for all real α. Therefore $P(t, s)$ is measurable on I^2.

From (1.15) it follows that $P(t, t)$ is measurable on I. Similarly, $N(t, s)$ is measurable on I^2 and $N(t, t)$ is measurable on I. Since $M(t, 0)$ and $M(0, s)$ are measurable on I, it follows from (1.14) that $M(t, s)$ is measurable on I^2 and $M(t, t)$ is measurable on I.

Because of (1.15), $P(t, t)$ is continuous almost everywhere on I. Therefore, from (1.16) and (1.17) it follows that for almost all $t \in I$,

$$\lim_{s \to t^+} P(t, s) = P(t, t).$$

Similarly, for almost all $t \in I$,

$$\lim_{s \to t^+} N(t, s) = N(t, t).$$

Since $M(0, s)$ is of B.V. on I,

$$\lim_{s \to t^+} M(0, s) = M(0, t)$$

for almost all $t \in I$. Therefore (1.12) follows from (1.14).

Let $\epsilon > 0$ and define

$$P^\epsilon(t, s) = \frac{1}{\epsilon} \int_s^{s+\epsilon} P(t, v) dv, \qquad (t, s) \in I^2,$$

$$N^\epsilon(t, s) = \frac{1}{\epsilon} \int_s^{s+\epsilon} P(t, v) dv, \qquad (t, s) \in I^2,$$

where $P(t, s) = P(t, 1)$ and $N(t, s) = N(t, 1)$ if $s > 1$. Then from (1.14),

$$M^\epsilon(t, s) = P^\epsilon(t, s) - N^\epsilon(t, s) + M^\epsilon(0, s) + M(t, 0) - M(0, 0)$$

and

$$\text{(1.18)} \qquad \operatorname*{var}_{(t,s) \in I^2} M^\epsilon(t, s) < \operatorname*{var}_{(t,s) \in I^2} P^\epsilon(t, s) + \operatorname*{var}_{(t,s) \in I^2} N^\epsilon(t, s).$$

Let $0 \leqq t' \leqq t'' \leqq 1$ and $0 \leqq s' \leqq s'' \leqq 1$. Then

$$P^\epsilon(t'', s'') - P^\epsilon(t', s'') - P^\epsilon(t'', s') + P^\epsilon(t', s')$$
$$= \frac{1}{\epsilon} \int_{s''}^{s''+\epsilon} [P(t'', v) - P(t', v) - P(t'', v - (s'' - s')) + P(t', v - (s'' - s'))] dv$$
$$\geqq 0$$

since

$$P(t'', v) - P(t', v) - P(t'', v - (s'' - s')) + P(t', v - (s'' - s')) \geqq 0$$

for all v, $s'' \leqq v \leqq s'' + \epsilon$ as can be seen from the definition. It follows that

$$\begin{aligned}\operatorname*{var}_{(t,s)\in I^2} P^\epsilon(t, s) &= P^\epsilon(1, 1) - P^\epsilon(1, 0) - P^\epsilon(0, 1) + P^\epsilon(0, 0)\\ &\leqq P^\epsilon(1, 1) - P^\epsilon(1, 0)\\ &\leqq \frac{1}{\epsilon}\int_1^{1+\epsilon} P(1, 1)dv \leqq P(1, 1)\end{aligned}$$

independent of $\epsilon > 0$. Similarly,

$$\operatorname*{var}_{(t,s)\in I^2} N(t, s) \leqq N(1, 1)$$

independent of $\epsilon > 0$. Therefore (1.13) follows from (1.18).

LEMMA 5. *Let $M(t, s)$ be of B.V. on I^2 and $J(t)$ be of B.V. on I. Let $\overline{M}(t, s)$, $M^\epsilon(t, s)$, and $(M^\epsilon)^*$ (t, s) be defined by* (0.7), (0.20), (0.21), *and* (0.22). *Then*

(1.19) $$\lim_{\epsilon\to 0^+} D(M^\epsilon)^* = D(\overline{M}).$$

Proof. We will show that $M^\epsilon(t, s)$ is of B.V. on I^2. Since $M^\epsilon(t, 1) = 0$, $t \in I$, and $M^\epsilon(0, s)$ is absolutely continuous on I, it follows from Lemma 4 that $M^\epsilon(t, s)$ is of B.V. on I^2. Since $J^\epsilon(t)$ is measurable, we conclude from Lemma 4 that $(M^\epsilon)^*(t, s)$ is measurable on I^2 and $(M^\epsilon)^*(t, s)$ is measurable on I. Since $M(t, s)$ is bounded on I^2, it follows that $M^\epsilon(t, s)$ is bounded independent of $\epsilon > 0$. Since $J(t)$ is bounded, $(M^\epsilon)^*(t, s)$ is bounded independent of $\epsilon > 0$. From Lemma 4 we know that $M(t, s)$ is of B.V. in s on I for all $t \in I$. Therefore

$$\lim_{\epsilon\to 0^+} M^\epsilon(t, s) = M(t, s)$$

almost everywhere on I^2. From conclusion (1.12) of Lemma 4 it follows that

$$\lim_{\epsilon\to 0^+} M^\epsilon(t, t) = M(t, t)$$

almost everywhere on I. Since $J(t)$ is of B.V. we have

$$\lim_{\epsilon\to 0^+} J^\epsilon(t) = J(t)$$

almost everywhere on I. Therefore

$$\lim_{\epsilon\to 0^+} (M^\epsilon)^*(t, s) = \overline{M}(t, s)$$

almost everywhere on I^2 and

$$\lim_{\epsilon\to 0^+} (M^\epsilon)^*(t, t) = \overline{M}(t, t)$$

almost everywhere on I. Therefore (1.19) follows from conclusion (1.1) of Lemma 1.

2. **Proof of Theorem C.** We make the following manipulations so that we can use Theorem B.

$$\int_0^1\int_0^t (M^\epsilon)^*(u,s)dudx(s) = \int_0^1\int_0^t M^\epsilon(u,s)dudx(s) + \int_0^t\int_s^t J^\epsilon(u)dudx(s)$$

$$= \int_0^1\int_0^t \frac{1}{\epsilon}\int_s^{s+\epsilon} M(u,v)dvdudx(s) - \int_0^t x(s)d_s\int_s^t J^\epsilon(u)du$$

(2.1)

$$= -\int_0^1 x(s)d_s \frac{1}{\epsilon}\int_s^{s+\epsilon}\int_0^t M(u,v)dudv + \int_0^t x(s)J^\epsilon(s)ds$$

$$= \int_0^1\left[-\int_0^t \frac{1}{\epsilon}[M(u,s+\epsilon) - M(u,s)]du\right]x(s)ds + \int_0^t J^\epsilon(s)x(s)ds.$$

Now for fixed $\epsilon>0$,

$$-\frac{1}{\epsilon}[M(t,s+\epsilon) - M(t,s)]$$

satisfies (0.12) and satisfies (0.14) because $M(t, s)$ is continuous on $0<s<\infty$ for all $t\in I$. $J^\epsilon(t)$ satisfies (0.13). From Lemma 5 and condition (0.9) we know there exists $\delta_1>0$ such that for $0<\epsilon<\delta_1$,

$$D((M^\epsilon)^*(t,s)) \neq 0.$$

Therefore, from (1.4) and conclusion (1.5) of Lemma 3, condition (0.15) of Theorem B is satisfied when $0<\epsilon<\delta_1$. Therefore we can write (0.17) with $H(t, s)$ replaced by $-(M(t, s+\epsilon)-M(t, s))/\epsilon$ and $J(t)$ replaced by $J^\epsilon(t)$ and we have by (1.5), (0.17), and (2.1), if $f(y)$ is measurable on C and $0<\epsilon<\delta_1$,

$$\int_c^w F(y)d_wy = |D((M^\epsilon)^*(t,s))| \int_c^w F\left(x(\cdot) + \int_0^1\int_0^{(\cdot)} (M^\epsilon)^*(u,s)dudx(s)\right)$$

$$\cdot\exp\left\{-\int_0^1\left\{\int_0^1\left[-\frac{1}{\epsilon}[M(t,s+\epsilon) - M(t,s)]\right]x(s)ds + J^\epsilon(t)x(t)\right\}^2 dt\right.$$

$$-2\int_0^1\int_0^1 -\frac{1}{\epsilon}[M(t,s+\epsilon) - M(t,s)]x(s)dsdx(t)$$

$$\left.-\int_0^1 J^\epsilon(t)d(x(t))^2\right\} d_wx$$

$$= |D(M^\epsilon)^*| \int_c^w F\left(x(\cdot) + \int_0^1\int_0^{(\cdot)} (M^\epsilon)^*(u,s)dudx(s)\right)$$

$$\cdot\exp\left\{-\int_0^1\left\{\int_0^1 (M^\epsilon)^*(t,s)dx(s)\right\}^2 dt - 2\int_0^1\int_0^1 M^\epsilon(t,s)dx(s)dx(t)\right.$$

$$\left.-\int_0^1 J^\epsilon(t)d(x(t))^2\right\} d_wx,$$

provided either side exists. This proves Theorem C.

3. In this section we prove some lemmas which we use in letting $\epsilon \to 0^+$.

LEMMA 6. *Let $N(t, s)$ be measurable in t on I for every $s \in I$ and let*

$$|N(t, s)| < B < \infty$$

and

$$\operatorname*{var}_{s \in I} N(t, s) < B < \infty,$$

where B is independent of $(t, s) \in I^2$. Then if $x \in C$,

$$\int_0^1 \int_0^t N(u, s) du dx(s)$$

exists and equals

$$\int_0^t \int_0^1 N(u, s) dx(s) du$$

for all $t \in I$.

Proof. For every $u \in I$, we have

$$\lim_{\text{norm } P \to 0^+} \sum_{i=1}^{n} N(u, s_i)[x(s_i) - x(s_{i-1})] = \int_0^1 N(u, s) dx(s)$$

where P is the partition

$$0 = s_0 \leqq \cdots \leqq s_n = 1.$$

From the hypothesis it follows that the limitand on the left is measurable and bounded independent of the partition. Then from the bounded convergence theorem, we have

$$\int_0^1 \int_0^t N(u, s) du dx(s)$$

exists and equals

$$\int_0^t \int_0^1 N(u, s) dx(s) du.$$

LEMMA 7. *Let $M(t, s)$ be of B.V. on I^2 and $J(t)$ be of B.V. on I. Let $(M^\epsilon)^*(t, s)$ and $\overline{M}(t, s)$ be defined by (0.22) and (0.7) respectively. Then*

$$\int_0^1 (M^\epsilon)^*(t, s) dx(s)$$

converges in Hilbert norm on $C \otimes I$ to

$$\int_0^1 \overline{M}(t, s)dx(s)$$

as $\epsilon \to 0^+$.

Proof. By a theorem of [2] and Fubini's theorem,

$$\int_{c\otimes I}^{w} \left(\int_0^1 [(M^\epsilon)^*(t, s) - M(t, s)]dx(s) \right)^2 d_w(x \otimes t)$$
$$= \int_0^1 \frac{1}{2} \int_0^1 [(M^\epsilon)^*(t, s) - \overline{M}(t, s)]^2 ds dt$$
$$= \frac{1}{2} \int_{I^2} [(M^\epsilon)^*(t, s) - \overline{M}(t, s)]^2 d(s \otimes t).$$

The integrand on the right is bounded and tends to zero almost everywhere on I^2 as ϵ tends to 0^+ because for every $t \in I$, $M(t, s)$ is a continuous function of s almost everywhere on I. The conclusion follows by bounded convergence.

LEMMA 8. *Let* $M(t, s)$ *be of B.V. on* I^2, $M(t, 1) = M(1, t) = 0$, $t \in I$, *and* $x \in C$. *Then*

$$\int_0^1 \int_0^1 M(t, s)dx(s)dx(t)$$

exists and equals

$$\int_{I^2} x(t)x(s)dM(t, s).$$

Proof. We need to show that

$$\int_0^1 M(t, s)dx(s)$$

is of B.V. Let $0 \leqq t_0 \leqq \cdots \leqq t_n \leqq 1$ be a partition of I. Then $M(t, 1) = 0$, $t \in I$, and we have

$$\sum_{i=1}^{n} \left| \int_0^1 M(t_i, s)dx(s) - \int_0^1 M(t_{i-1}, s)dx(s) \right|$$
$$(3.1) \qquad = \sum_{i=1}^{n} \left| \int_0^1 x(s)d[M(t_i, s) - M(t_{i-1}, s)] \right|$$
$$\leqq |||x||| \sum_{i=1}^{n} \operatorname*{var}_{s\in I} [M(t_i, s) - M(t_{i-1}, s)].$$

$M(t_i, s)$ is of B.V. in s on I for each $i = 0, 1, \cdots, n$, by Lemma 4. Therefore, we can find n partitions $\{0 \leqq s^{(i)} \leqq \cdots \leqq s^{(i)}_{m_i} \leqq 1\}_{i=1}^n$ such that

$$\operatorname*{var}_{s \in I} [M(t_i, s) - M(t_{i-1}, s)] < \frac{1}{n} + \sum_{j=1}^{m_i} \left| M(t_i, s_j^{(i)}) - M(t_{i-1}, s_j^{(i)}) - M(t_i, s_{j-1}^{(i)}) - M(t_{i-1}, s_{j-1}^{(i)}) \right|,$$

$i = 1, 2, \cdots, n$. Then if $0 \leqq s_0 \leqq \cdots \leqq s_m \leqq 1$ is the union of the partitions $\{0 \leqq s^{(i)} \leqq \cdots \leqq s^{(i)}_{m_i} \leqq 1\}_{i=1}^n$,

$$\begin{aligned} \sum_{i=1}^n \operatorname*{var}_{s \in I} [M(t_i, s) - M(t_{i-1}, s)] & \\ \leqq 1 + \sum_{i=1}^n \sum_{j=1}^m \left| M(t_i, s_j) - M(t_{i-1}, s_j) - M(t_i, s_{j-1}) + M(t_{i-1}, s_{j-1}) \right| & \quad (3.2) \\ \leqq 1 + \operatorname*{var}_{(t,s) \in I^2} M(t, s). & \end{aligned}$$

Now (3.1) and (3.2) show that $\int_0^1 M(t, s)dx(s)$ is of B.V. Therefore

$$\int_0^1 \int_0^1 M(t, s)dx(s)dx(t)$$

exists.

Let $\epsilon > 0$ be given. Since $x \in C$ and

$$\operatorname*{var}_{(t,s) \in I^2} M(t, s) < \infty,$$

$$\int_{I^2} x(t)x(s)dM(t, s)$$

exists. Since $M(1, s) = M(t, 1) = 0$,

$$\int_{I^2} x(t)x(s)dM(t, s) = \int_{I^2} M(t, s)d(x(t)x(s)). \tag{3.3}$$

Choose δ such that if the partitions T, $0 = t_0 \leqq \cdots \leqq t_n = 1$, and S, $0 = s_0 \leqq \cdots \leqq s_m = 1$, have norm less than δ, then

$$\left| \int_{I^2} M(t, s)d(x(t)x(s)) - \sum_{i=1}^n \sum_{j=1}^m M(t_i, s_j)(x(s_j) - x(s_{j-1}))(x(t_i) - x(t_{i-1})) \right| < \frac{\epsilon}{3} \tag{3.4}$$

and

$$\left| \int_0^1 \int_0^1 M(t, s)dx(s)dx(t) - \sum_{i=1}^n \int_0^1 M(t_i, s)dx(s)(x(t_i) - x(t_{i-1})) \right| < \frac{\epsilon}{3} \cdot \tag{3.5}$$

Let T be fixed with norm $T<\delta$. Choose δ' such that if norm $S<\delta'$, then for $i=1, 2, \cdots, n$,

$$(3.6)\quad \left|\int_0^1 M(t_i, s)dx(s) - \sum_{j=1}^{m} M(t_i, s_j)(x(s_j) - x(s_{j-1}))\right| < \frac{\epsilon}{6n\,|||x|||+1}.$$

Let norm $S<\min(\delta, \delta')$. Using (3.4)–(3.6) we obtain

$$\left|\int_{I^2} M(t, s)d(x(t)x(s)) - \int_0^1\int_0^1 M(t, s)dx(s)dx(t)\right| < \epsilon.$$

Since ϵ is arbitrary, we have

$$\int_{I^2} M(t, s)d(x(t)x(s)) = \int_0^1\int_0^1 M(t, s)dx(s)dx(t).$$

Therefore the conclusion follows from this and (3.3).

LEMMA 9. *Let $M(t, s)$ be of B.V. on I^2 and $M(t, 1)=M(1, s)=0$ and let $M^\epsilon(t, s)$ be defined as in* (0.20). *Then*

$$\operatorname*{l.i.m.}_{\epsilon\to 0^+} \int_0^1\int_0^1 M^\epsilon(t, s)dx(s)dx(t) = \int_0^1\int_0^1 M(t, s)dx(s)dx(t)$$

over C.

Proof. Let $N_\epsilon(t, s)=M^\epsilon(t, s)-M(t, s)$. By Lemma 4, $N_\epsilon(t, s)$ is of B.V. on I^2. Also $N_\epsilon(t, 1)=N_\epsilon(1, s)=0$. By Lemma 8,

$$\begin{aligned}\left[\int_0^1\int_0^1 N_\epsilon(t, s)dx(s)dx(t)\right]^2 &= \left[\int_{I^2} x(t)x(s)dN_\epsilon(t, s)\right]^2\\ (3.7)\qquad &= \int_{I^2}\int_{I^2} x(t)x(s)x(u)x(v)dN_\epsilon(t, s)dN_\epsilon(u, v)\\ &= \int_{I^2\otimes I^2} x(t)x(s)x(u)x(v)d(N_\epsilon(u, v)N_\epsilon(t, s))\end{aligned}$$

from the Fubini theorem. Also from the Fubini theorem,

$$\begin{aligned}&\int_C^w\left(\int_0^1\int_0^1 N_\epsilon(t, s)dx(s)dx(t)\right)^2 d_wx\\ (3.7)\qquad &= \int_C^w\int_{I^2\otimes I^2} x(t)x(s)x(u)x(v)d(N_\epsilon(u, v)N_\epsilon(t, s))d_wx\\ &= \int_{I^2\otimes I^2} U(t, s, u, v)d(N_\epsilon(u, v)N_\epsilon(t, s)),\end{aligned}$$

where

$$U(t, s, u, v) = \int_C^w x(t)x(s)x(u)x(v)dx, \qquad (t, s, u, v) \in I^2 \otimes I^2.$$

The use of the Fubini theorem is justified since $|||x|||^4$ is integrable on C. Since if $0 \leqq t_1 \leqq t_2 \leqq t_3 \leqq t_4$,

$$\int_C^w x(t_1)x(t_2)x(t_3)x(t_4)d_w x = t_1\left(\frac{t_2}{2} + \frac{t_3}{4}\right),$$

$U(t, s, u, v)$ is absolutely continuous on $I^2 \otimes I^2$. Since $U(t, s, u, v)$ is zero if any argument is zero and $N_\epsilon(t, 1)$ and $N_\epsilon(1, s)$ are zero, we have on integrating by parts,

$$(3.8) \quad \int_{I^2 \otimes I^2} U(t, s, u, v)d(N_\epsilon(u, v)N_\epsilon(t, s)) = \int_{I^2 \otimes I^2} N_\epsilon(t, s)N_\epsilon(u, v)dU(t, s, u, v).$$

For every fixed $t \in I$, $M(t, s)$ is continuous in s almost everywhere on I. Therefore $N_\epsilon(t, s)$ converges almost everywhere to zero as $\epsilon \rightarrow 0^+$. Since $N_\epsilon(t, s)$ is bounded on I^2 and $U(t, s, u, v)$ is absolutely continuous, the right hand side of (3.8) tends to zero as $\epsilon \rightarrow 0^+$. Using (3.7) the lemma is established.

LEMMA 10. *Let $J(t)$ be of B.V. on I and $J(1) = 0$. Let $J^\epsilon(t)$ be defined by* (0.21). *Then*

$$\underset{\epsilon \rightarrow 0^+}{\text{l.i.m.}} \int_0^1 J^\epsilon(t)d(x(t))^2 = \int_0^1 J(t)d(x(t))^2 \text{ over } C.$$

Proof. The proof is similar to the proof of Lemma 9. Let $N_\epsilon(t) = J^\epsilon(t) - J(t)$. By using Lemma 8 and the Fubini theorem,

$$\begin{aligned} \int_C^w \left(\int_0^1 N_\epsilon(t)d(x(t))^2\right)^2 d_w x &= \int_C^w \left(\int_0^1 \int_0^1 N_\epsilon(t)N_\epsilon(s)d(x(t))^2 d(x(s))^2\right) d_w x \\ &= \int_C^w \int_{I^2} (x(t)x(s))^2 d(N_\epsilon(t)N_\epsilon(s))d_w x \\ &= \int_{I^2} \chi(t, s)d(N_\epsilon(t)N_\epsilon(s)) \\ &= \int_{I^2} N_\epsilon(t)N_\epsilon(s)d\chi(t, s), \end{aligned}$$

where $\chi(t, s) = \int_C^w (x(t)x(s))^2 d_w x$ is absolutely continuous. The conclusion follows by bounded convergence since $J(t)$ is bounded.

4. **Proof of Theorem A.** We assume condition (0.19) temporarily. Let $F(y)$ be bounded and continuous in the uniform topology and vanish if $|||y||| \geqq R$.

From Lemmas 7, 9, and 10, there exists a sequence $\{\epsilon_n\}$ such that

$$\lim_{n\to\infty} \epsilon_n = 0$$

and

$$\lim_{n\to\infty} \int_0^1 (M^{\epsilon_n})^*(t, s)dx(s) = \int_0^1 \overline{M}(t, s)dx(s) \tag{4.1}$$

almost everywhere on $C \otimes I$,

$$\lim_{n\to\infty} \int_0^1 \int_0^1 M^{\epsilon_n}(t, s)dx(s)dx(t) = \int_0^1 \int_0^1 M(t, s)dx(s)dx(t) \tag{4.2}$$

almost everywhere on C, and

$$\lim_{n\to\infty} \int_0^1 J^{\epsilon_n}(t)d(x(t))^2 = \int_0^1 J(t)d(x(t))^2 \tag{4.3}$$

almost everywhere on C.

Consider

$$\begin{aligned} &\left| \int_0^t \int_0^1 (M^{\epsilon_n})^*(u, s)dx(s)du - \int_0^t \int_0^1 \overline{M}(u, s)dx(s)du \right| \\ &\qquad \leqq \int_0^1 \left| \int_0^1 (M^{\epsilon_n})^*(u, s)dx(s) - \int_0^1 \overline{M}(u, s)dx(s) \right| du. \end{aligned} \tag{4.4}$$

Since by Lemma 4

$$\operatorname*{var}_{(t,s)\in I^2} M^{\epsilon}(t, s)$$

is bounded independent of ϵ and since $M^{\epsilon}(1, s) = 0$ and $J^{\epsilon}(t)$ is bounded independent of ϵ and t,

$$\operatorname*{var}_{s\in I} (M^{\epsilon})^*(t, s)$$

is bounded independent of t and ϵ. Therefore the integrand on the right of (4.4) is bounded on I independent of $\epsilon > 0$ for every $x \in C$. By (4.1) this integrand tends to zero as n tends to ∞ for almost all u on I for almost all $x \in C$. It follows from (4.4), Lemma 6, and bounded convergence that

$$\lim_{n\to\infty} \int_0^1 \int_0^t (M^{\epsilon_n})^*(u, s)dudx(s) = \int_0^1 \int_0^t \overline{M}(u, s)dudx(s) \tag{4.5}$$

uniformly in t on I for almost all $x \in C$.

From Lemma 4 it follows that

$$\int_0^1 M^\epsilon(t, s)dx(s)$$

is bounded on I independent of ϵ for every $x \in C$. Since $J^\epsilon(t)$ is bounded on I independent of ϵ, we have from (4.1) and bounded convergence,

$$\lim_{n\to\infty} \int_0^1 \left[\int_0^1 (M^{\epsilon_n})^*(t, s)dx(s)\right]^2 dt = \int_0^1 \left(\int_0^1 \overline{M}(t, s)dx(s)\right)^2 dt \tag{4.6}$$

for almost every $x \in C$.

We will now show that there exists R_1 independent of n such that for sufficiently large n, $n > N$,

$$F\left(x(\cdot) + \int_0^1 \int_0^{(\cdot)} (M^{\epsilon_n})^*(u, s)dudx(s)\right)$$

vanishes if

$$|||x||| \geqq R_1.$$

From (1.19) of Lemma 5,

$$\lim_{\epsilon\to 0^+} D(M^\epsilon)^* = D(\overline{M}). \tag{4.7}$$

From (0.9) there exists N such that if $n > N$, $D(M^{\epsilon_n})^*$ is bounded from zero. We shall assume from now on that $n > N$. From [3] we know there exists the Volterra reciprocal kernel $((M^{\epsilon_n})^*)^{-1}(t, s)$ such that

$$\begin{aligned} ((M^{\epsilon_n})^*)^{-1}(t, s) + (M^{\epsilon_n})^*(t, s) &= -\int_0^1 ((M^{\epsilon_n})^*)^{-1}(t, u)(M^{\epsilon_n})^*(u, s)du \\ &= -\int_0^1 (M^{\epsilon_n})(t, u)((M^{\epsilon_n})^*)^{-1}(u, s)du, \end{aligned} \tag{4.8}$$

and $((M^{\epsilon_n})^*)^{-1}(t, s)$ is bounded on I^2 independent of n. From (4.8) and Lemma 4, (1.13),

$$\begin{aligned} &\operatorname*{var}_{s\in I} (M^{\epsilon_n})^*)^{-1}(t, s) \\ &\quad < \operatorname*{var}_{s\in I} (M^{\epsilon_n})^*(t, s) + \operatorname*{var}_{s\in I} \int_0^1 ((M^{\epsilon_n})^*)^{-1}(t, u)(M^{\epsilon_n})^*(u, s)du \\ &\quad < \operatorname*{var}_{s\in I} (M^{\epsilon_n})^*(t, s)\left[1 + \sup_{(t,s)\in I^2} \left| ((M^{\epsilon_n})^*)^{-1}(t, s) \right|\right] \\ &\quad < B, \end{aligned} \tag{4.9}$$

where B is independent of t and n. Therefore from (4.9) and Lemma 6 which we use by virtue of (4.8) and (4.9), if $|||y||| < R$, then

$$\left|\left|\left| y(t) + \int_0^1 \int_0^t ((M^{\epsilon_n})^*)^{-1}(u, s)dudy(s) \right|\right|\right| \tag{4.10}$$

$$\leqq R\left[1 + B + \sup_{t\in I} |((M^{\epsilon_n})^*)^{-1}(t, 1)|\right] < R_1$$

which is bounded independent of n.

Consider

$$y(t) = x(t) + \int_0^1 \int_0^t (M^{\epsilon_n})^*(u, s)dudx(s). \tag{4.11}$$

Then by Lemma 6 and (4.8),

$$y(t) + \int_0^1 \int_0^t ((M^{\epsilon_n})^*)^{-1}(u, s)dudy(s)$$

$$= x(t) + \int_0^1 \int_0^t (M^{\epsilon_n})^*(u, s)dudx(s) + \int_0^1 \int_0^t ((M^{\epsilon_n})^*)^{-1}(u, s)dudx(s)$$

$$+ \int_0^1 \int_0^t ((M^{\epsilon_n})^*)^{-1}(u, s)dud_s \int_0^1 \int_0^s (M^{\epsilon_n})^*(w, v)dwdx(v)$$

$$= x(t) + \int_0^1 \int_0^t [(M^{\epsilon_n})^*(u, s) + ((M^{\epsilon_n})^*)^{-1}(u, s)]dudx(s) \tag{4.12}$$

$$+ \int_0^1 \int_0^t ((M^{\epsilon_n})^*)^{-1}(u, s)du \int_0^1 (M^{\epsilon_n})^*(s, v)dx(v)ds$$

$$= x(t) + \int_0^1 \int_0^t [(M^{\epsilon_n})^*(u, v) + ((M^{\epsilon_n})^*)^{-1}(u, v)dudx(v)$$

$$+ \int_0^1 \int_0^t \left[\int_0^1 ((M^{\epsilon_n})^*)^{-1}(u, s)(M^{\epsilon_n})^*(s, v)ds\right]dudx(v)$$

$$= x(t).$$

From (4.10)–(4.12) it follows that there exists R_1 independent of n such that if

$$|||x||| > R_1,$$

then

$$\left|\left|\left| x(t) + \int_0^1 \int_0^t (M^{\epsilon_n})^*(u, s)dudx(s) \right|\right|\right| \geqq R.$$

Therefore

$$F\left(x(\cdot) + \int_0^1 \int_0^{(\cdot)} (M^{\epsilon_n})^*(u, s)dudx(s)\right)$$

vanishes if $|||x||| > R_1$.

From Lemmas 4 and 8

$$\int_0^1 \int_0^1 M^{\epsilon_n}(t, s)dx(s)dx(t)$$

is bounded independent of n and $x \in C$ as long as $|||x||| < R_1$. Also

$$\int_0^1 J^{\epsilon_n}(t)d(x(t))^2$$

is bounded independent of n and $x \in C$ as long as $|||x||| < R_1$ because

$$\operatorname*{var}_{t \in I} J^{\epsilon}(t)$$

is bounded independent of $\epsilon > 0$. Because

$$F\left(x(\cdot) + \int_0^1 \int_0^{(\cdot)} (M^{\epsilon_n})^*(u, s)dudx(s)\right)$$

vanishes if $|||x||| \geqq R_1$ and is bounded and $F(y)$ is continuous in the uniform topology and because of (4.5), (4.6), (4.7), (4.2) and (4.3) we have, if (0.23) holds in some range $0 < \epsilon < \delta$, by bounded convergence,

$$\begin{aligned} \int_C^w F(y)d_wy = \; | D(\overline{M}) | \int_C^w F\left(x(\cdot) + \int_0^1 \int_0^{(\cdot)} \overline{M}(u, s)dudx(s)\right) \\ \cdot \exp\left\{ - \int_0^1 \left(\int_0^1 \overline{M}(t, s)dx(s)\right)^2 dt \right. \\ \left. - 2\int_0^1 \int_0^1 M(t, s)dx(s)dx(t) - \int_0^1 J(t)d(x(t))^2 \right\} d_wx. \end{aligned} \tag{4.13}$$

Therefore, under the hypothesis of Theorem A, the restrictions on $F(y)$, and condition (0.19), if (0.23) holds in a range $0 < \epsilon < \delta$, then (4.13) holds. By Lemmas 4 and 5, there exists δ_2 such that if $0 < \eta < \delta_2$ the hypothesis of Theorem C is satisfied when $M(t, s)$ and $J(t)$ are replaced by $M^\eta(t, s)$ and $J^\eta(t)$ respectively. Consequently there exists $\delta_3 > 0$ such that (0.23) holds for $0 < \epsilon < \delta_3$ and $0 < \eta < \delta_2$ with $M(t, s)$ and $J(t)$ replaced by $M^\eta(t, s)$ and $J^\eta(t)$ respectively so that (4.13) holds with $M(t, s)$ and $J(t)$ replaced by $M^\eta(t, s)$ and $J^\eta(t)$ respectively for $0 < \eta < \delta_2$. Therefore (4.13) holds. We may remove condition (0.19) since the values of $J(1)$, $M(t, 1)$, and $M(1, s)$ do not affect the formula (4.13) provided the hypothesis of Theorem A still holds.

From (0.4), (0.5) and (0.6) it follows that

$$L(t, s) = \int_0^t \overline{M}(u, s)du \tag{4.14}$$

for all $(t, s) \in I^2$. Therefore (0.10) follows from (4.13) when $F(y)$ is continuous in the uniform topology, is bounded, and vanishes outside a uniform sphere. We can progressively enlarge the class of functionals for which (0.10) holds to the class of all measurable functionals as done in [1, p. 215].

Since $D(\overline{M}) \neq 0$ there is a Volterra reciprocal kernel, $\overline{M}^{-1}(t, s)$, such that

$$
\begin{aligned}
\overline{M}^{-1}(t, s) + M(t, s) &= -\int_0^1 \overline{M}^{-1}(t, u)\overline{M}(u, s)du \\
&= -\int_0^1 \overline{M}(t, u)\overline{M}^{-1}(u, s)du.
\end{aligned} \tag{4.15}
$$

Proceeding very much as in (4.12) one can show with the aid of (4.15) and (4.14) that the transformation (0.1) carries C onto C in a one to one manner.

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